Disjoint Simplices and Geometric Hypergraphs

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INTRODUCTION

Let A be a set of $2n$ points in general position in the Euclidean plane R^2, and suppose n of the points are colored red and the remaining n are colored blue. A celebrated Putnam problem (see [6]) asserts that there are n pairwise disjoint straight line segments matching the red points to the blue points. To show this, consider the set of all $n!$ possible matchings and choose one, M, that minimizes the sum of lengths $l(M)$ of its line segments. It is easy to show that these line segments cannot intersect. Indeed, if the two segments v_1, b_1 and v_2, b_2 intersect, where v_1, v_2 are two red points and b_1, b_2 are two blue points, the matching M' obtained from M by replacing $v_1 b_1$ and $v_2 b_2$ by $v_1 b_2$ and $v_2 b_1$ satisfies $l(M') < l(M)$, contradicting the choice of M. Our first result in this paper is a generalization of this result to higher dimensions.

THEOREM 1: Let A be a set of $d \cdot n$ points in general position in R^d, and let $A = A_1 \cup A_2 \cup \cdots \cup A_d$ be a partition of A into d pairwise disjoint sets, each consisting of n points. Then there are n pairwise disjoint $(d-1)$-dimensional simplices, each containing precisely one vertex from each A_i, $1 \leq i \leq d$.

We prove this theorem in the next section. The proof is short but uses a nonelementary tool: the well-known Borsuk-Ulam theorem.

Combining Theorem 1 with an old result of Erdös from extremal graph theory we obtain a corollary dealing with geometric hypergraphs. A *geometric d-hypergraph* is a pair $G = (V, E)$, where V is a set of points called vertices, in general position in R^d, and E is a set of (closed) $(d-1)$-dimensional simplices called edges, whose vertices are points of V. If $d = 2$, G is called a *geometric graph.* It is well known (see [3], [5]) that every geometric graph with n vertices and $n + 1$ edges contains two disjoint edges, two nonintersecting edges, and this result is the best possible. The number of edges that guarantees l pairwise disjoint edges is not known for $l > 2$, although Perles [7] determined the exact number for the case that the set of vertices

V is the set of vertices of a convex polygon. The situation seems much more difficult for geometric d-hypergraphs, when $d > 2$. Even the number of edges that guarantees two disjoint simplices is not known in this case. Clearly this number is greater than $\binom{n-1}{d-1}$ (simply take all edges containing a given point) and is at most $\binom{n}{d}$. In the final section we prove the following theorem, that implies that for every fixed d, $l \geq 2$, every geometric d-hypergraph on n vertices that contains no l pairwise nonintersecting edges has $o(n^d)$ edges.

THEOREM 2: Every geometric d-hypergraph with n vertices and at least $n^{d-(1/l^{d-1})}$ edges contains l pairwise nonintersecting edges.

It is worth noting that the following, much stronger conjecture seems plausible.

CONJECTURE 1: For every l, $d \geq 2$ there exists a constant $c = c(l, d)$ such that every geometric d-hypergraph with n vertices and at least $c \cdot n^{d-1}$ edges contains l pairwise nonintersecting edges.

We do not know how to prove this conjecture, even for $d = 2$, $l = 3$.

PROOF OF THEOREM 1

We need the following lemma, sometimes called the "Ham-Sandwich theorem," which is a well-known consequence of the Borsuk-Ulam theorem (see [1], [2]).

LEMMA 1: Let $\mu_1, \mu_2, \ldots, \mu_d$ be d continuous probability measures in R^d. Then there exists a hyperplane H in R^d that bisects each of the d measures, that is, $\mu_i(H^+) = \mu_i(H^-)(=\frac{1}{2})$ for all $1 \leq i \leq d$, where H^+ and H^- denote, respectively, the open positive side and the open negative side of H.

Theorem 1 will be derived from the following lemma.

LEMMA 2: Let $A, A_1, A_2, \ldots, A_d$ be as in Theorem 1. Then there exists a hyperplane H in R^d such that

$$|H^+ \cap A_i| = [n/2] \quad \text{and} \quad |H^- \cap A_i| = [n/2] \quad \text{for all } 1 \leq i \leq d. \tag{1}$$

(Notice that if n is odd (1) implies that H contains precisely one point from each A_i.)

Proof: Replace each point $p \in A$ by a ball of radius ε centered in p, where ε is small enough to guarantee that no hyperplane intersects more than d balls. Associate each ball with a uniformly distributed measure of $1/n$. For $1 \leq i \leq d$ and a (lebesgue)-measurable subset T of R^d, define $\mu_i(T)$ as the total measure of balls centered at point of A_i captured by T. Clearly $\mu_1, \mu_2, \ldots, \mu_d$ are a continuous probability measure. By Lemma 1 there exists a hyperplane H in R^d such that $\mu_i(H^+) = \mu_i(H^-) = \frac{1}{2}$ for all $1 \leq i \leq d$. If n is odd, this implies that H intersects at least one ball centered at a point of A_i. However, H cannot intersect more than d balls altogether, and thus it intersects precisly one ball centered at a point of A_i, and it must bisect these d balls. Hence, for odd n, H satisfies (1). If n is even, H intersects at most d balls, and by slightly rotating H we can divide the centers of these balls between

H^+ and H^- as we wish, without changing the position of each other point of A with respect to H. One can easily check that this guarantees the existence of an H satisfying (1). □

We can now prove Theorem 1 by induction on n. For $n = 1$ the result is trivial. Assuming the result for all n', $n' < n$, let $A, A_1, A_2, \ldots, A_d$ be as in Theorem 1 and let H be a hyperplane, guaranteed by Lemma 2, satisfying (1). Put $B_i = H^+ \cap A_i$ and $C_i = H^- \cap A_i$ for $1 \le i \le d$, $B = B_i \cup \cdots \cup B_d$ and $C = C_1 \cup \cdots \cup C_d$. By applying the induction hypothesis to $B, B_1, \ldots, B_d$ and $C, C_1, \ldots, C_d$, we obtain two sets S_1 and S_2 of $[n/2]$ pairwise disjoint simplices each, where each simplex of S_1 contains precisly one vertex from each B_i and each simplex of S_2 contains precisely one vertex from each C_i. Clearly, all the simplices in S_1 lie in H^+ and all those in S_2 lie in H^-.

We thus obtained $2 \cdot [n/2]$ pairwise nonintersecting simplices. These, together with the simplex spanned by $A_i \cap H$ if n is odd, complete the induction and the proof of Theorem 1. □

PROOF OF THEOREM 2

We need the following result of Erdös.

LEMMA 3 [4]: Every d-uniform hypergraph with n vertices and at least $n^{d-(1/l^{d-1})}$ edges contains a complete d-partite subhypergraph on d classes of l vertices each.

Now suppose that G is a geometric d-hypergraph with n vertices and at least $n^{d-(1/l^{d-1})}$ edges. By Lemma 3 there is a set A of $l \cdot d$ vertices of G, $A = A_1 \cup \cdots \cup A_d$, where $|A_i| = l$ for each i, and all the l^d $(d-1)$-simplices consisting of one vertex from each A_i are edges of G. The assertion of Theorem 2 now follows from Theorem 1. □

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